Coordinate transformation makes perfect invisibility cloak with arbitrary shape

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Abstract. By investigating wave properties at cloak boundaries, invisibility cloaks with arbitrary shape constructed by general coordinate transformations are confirmed to be perfectly invisible to the external incident wave. The differences between line transformed cloaks and point transformed cloaks are discussed. The fields in the cloak medium are found analytically to be related to the fields in the original space via coordinate transformation functions. At the exterior boundary of the cloak, it is shown that no reflection is excited even though the permittivity and permeability do not always have a perfect matched layer form. While at the inner boundary, no reflection is excited either, and in particular no field can penetrate into the cloaked region. However, for the inner boundary of any line transformed cloak, the permittivity and permeability in a specific tangential direction are always required to be infinitely large. Furthermore, the field discontinuity at the inner boundary always exists; the surface current is induced to make this discontinuity self-consistent. For a point transformed cloak, it does not experience such problems. The tangential fields at the inner boundary are all zero, implying no field discontinuity exists.

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1. Introduction

The recent exciting development of invisibility cloaks, has attracted intense attentions and discussions [1]- [16]. Theoretically, the cloaks are constructed easily based on a coordinate transformation method as proposed in Ref. [1]. The object inside the cloak is invisible to the outside observer, because the light is excluded from the object and the exterior field is not perturbed. The invisibility of the linearly radially transformed cylindrical and spherical cloaks has been confirmed by both numerical calculations [3, 4] and analytical solutions [5, 6]. Experimentally, the invisibility cloak with simplified material parameters has been implemented by Schurig et.al at the microwave frequency. Inspired by the idea of the invisibility cloak, some interesting applications, such as field concentration [9], field rotation [10], and electromagnetic wormholes [11], have been proposed.

Up to now, most of discussions on invisibility cloaks focus on the cylindrical and spherical cloaks produced by a coordinate transformation only in the radial direction. For instance, in Ref. [1], linearly radially transformed cylindrical and spherical cloaks are discussed in detail, and their invisibility is confirmed by ray tracing. In Refs. [5] and [6], the invisibility performances of such cylindrical and spherical cloaks are further confirmed by obtaining the exact fields in the cloak medium directly from Maxwell's equations. In practice, it is sometimes desirable to have invisibility cloaks whose shapes are tailored for the objects to be cloaked. Thus, one needs to understand well about the properties of invisibility cloaks with arbitrary shape produced by general coordinate transformations. However, the investigations on an invisibility cloak with arbitrary shape are only seen in few papers [13,14]. The mechanism why the invisibility of a general cloak produced by compressing space is ensured, is still unclear. In this paper, we investigate the electromagnetic (EM) properties of invisibility cloaks with arbitrary shape constructed by general coordinate transformations, and we confirm their perfect invisibility. To figure out invisibility cloaks' main physical properties, we only focus on the ideal case without considering the practical implementation in this paper.

The paper is organized as follows. In section II, Maxwell's equations in a curved coordinate system are derived. In section III, we show how to construct an invisibility cloak by compressing space in a general manner. In section IV, the wave behaviors and the medium properties at the exterior boundary of the cloak are investigated. In section V, we study the the wave behaviors and the medium properties at the inner boundary of the cloak. Through sections IV and V, the invisibility of cloaks with arbitrary shape is confirmed, and the fields in the cloak medium are derived with simple expressions. In section VI, the cloak parameters and the fields in the cloak are derived when the transformed space is described under arbitrary coordinate system. In section VII, two examples of invisibility cloaks, i.e., cylindrical and spherical invisibility cloaks are investigated. In section VII, the paper is summarized.

2. Maxwell's equations in a curved coordinate system

Maxwell's Equations in a Cartesian (x, y, z) space take the form as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \quad \nabla \cdot \mathbf{D} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad (1)$$

with

$$\mathbf{D} = \epsilon_0 \bar{\bar{\varepsilon}} \cdot \mathbf{E}, \quad \mathbf{B} = \mu_0 \bar{\bar{\mu}} \cdot \mathbf{H}. \quad (2)$$

Consider the transformation from cartesian space to an arbitrary curved space described by coordinates q_1, q_2, q_3 with

$$x = f_1(q_1, q_2, q_3), \quad y = f_2(q_1, q_2, q_3), \quad z = f_3(q_1, q_2, q_3). \tag{3}$$

The length of a line element in the transformed space is given by $dl^2 = [dq_1, dq_2, dq_3]Q[dq_1, dq_2, dq_3]^T$, where the superscript "T" denotes the transpose of matrix, and $Q = gg^T$ with

$$g = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_2}{\partial q_1} & \frac{\partial f_3}{\partial q_1} \\ \frac{\partial f_1}{\partial q_2} & \frac{\partial f_2}{\partial q_2} & \frac{\partial f_3}{\partial q_2} \\ \frac{\partial f_1}{\partial q_3} & \frac{\partial f_2}{\partial q_3} & \frac{\partial f_3}{\partial q_3} \end{bmatrix}. \tag{4}$$

The volume of a space element is expressed as $dv = \det(g)dq_1 dq_2 dq_3$, where $\det(g)$ represents the determinant of g. Here, it is noted that the way of describing the space transformation in this paper is similar as in Ref. [13], where the time transformation is also taken into account. The space-time metric tensor $\mathbf{g}_{\alpha\beta}$ defined in Ref. [13] is $\mathbf{g}_{\alpha\beta} = diag[1, -Q]$ in the present paper, where only space transformation is considered.

Then Maxwell's equations in the curved space take the form as [1,13]

$$\nabla_q \times \widehat{\mathbf{E}} = -\frac{\partial \widehat{\mathbf{B}}}{\partial t}, \quad \nabla_q \times \widehat{\mathbf{H}} = \frac{\partial \widehat{\mathbf{D}}}{\partial t} + \widehat{\mathbf{j}}, \quad \nabla_q \cdot \widehat{\mathbf{D}} = \widehat{\rho}, \quad \nabla_q \cdot \widehat{\mathbf{B}} = 0 \tag{5}$$

with

$$\widehat{\mathbf{D}} = \epsilon_0 \widehat{\overline{\overline{\varepsilon}}} \cdot \widehat{\mathbf{E}}, \quad \widehat{\mathbf{B}} = \mu_0 \widehat{\overline{\overline{\mu}}} \cdot \widehat{\mathbf{H}}, \tag{6}$$

$$\widehat{\overline{\overline{\varepsilon}}} = \det(g)(g^T)^{-1}\overline{\overline{\varepsilon}}g^{-1}, \quad \widehat{\overline{\overline{\mu}}} = \det(g)(g^T)^{-1}\overline{\overline{\mu}}g^{-1}, \tag{7}$$

$$\widehat{\mathbf{j}} = \det(g)(g^T)^{-1}\mathbf{j}, \quad \widehat{\rho} = \det(g)\rho, \tag{8}$$

$$\widehat{\mathbf{E}} = g\mathbf{E}, \quad \widehat{\mathbf{H}} = g\mathbf{H}, \tag{9}$$

where the superscript "-1" denotes the inverse of matrix.

The permittivity and permeability $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ in the Cartesian space are considered for a general case, i.e., they can be tensors. It is seen above that Maxwell's equations in the curved space have the same form as in the Cartesian space. However, the definitions of the permittivity, permeability, current density, and electric charge density are different, as shown in Eqs. (7) and (8).

3. Construction of invisibility cloaks

To construct a cloak, one usually starts from compressing an enclosed space with the exterior boundary unchanged [1]. As seen in Fig. 1, the region enclosed by boundary S_1 is compressed to the region bounded by the exterior boundary S_1 and the interior boundary S_2. Such a space compression can be viewed as a certain coordinate transformation described by Eq. (3), which makes a connection between the points in the compressed space with coordinates (q_1, q_2, q_3) and the points with Cartesian coordinates (x, y, z) in the original space. The exterior boundary S_1 satisfies $q_1 = x, q_2 = y, q_3 = z$. Notice the interior boundary S_2 is obtained by blowing up a line or a point [15]. Thus the cloaks can be divided into two classes: line transformed cloaks and point transformed cloaks. The compressed shaded region in Fig. 1 is the desired cloak.

The permittivity and permeability tensors of the cloak in Cartesian coordinate system are given in Eq. (7). It seems that the cloak medium is very complex,
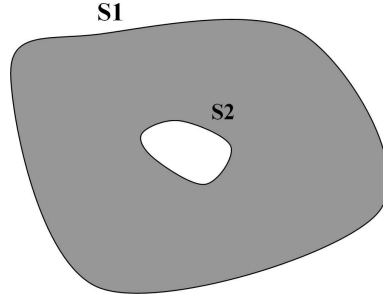
Figure 1. The cross section of the cloak (shaded region).

whose permittivity and permeability are tensors and their values vary with the spatial location. However, the eigen functions of the wave equations in the cloak medium are quite simple, which relate with the eigen function of the uncompressed space by Eq. (9).

In order to achieve the invisibility, the cloak should be able to exclude light from a protected object without perturbing the exterior field. Thus for the above cloak, at the exterior boundary S_1, external incident light should excite no reflection. While at the interior boundary S_2, no reflection is either excited, and light can't penetrate into the cloaked region. In the following sections, we will prove the invisibility of the cloak by investigating the wave behaviors at the cloak's exterior and inner boundaries. For the simplicity of our discussions and considering the practical application, the invisibility cloak is considered to be placed in air. Then the permittivity and permeability of the cloak in Eq. (7) will be simplified to

$$\widehat{\overline{\overline{\varepsilon}}} = \widehat{\overline{\overline{\mu}}} = \det(g)(g^T)^{-1}g^{-1}, \tag{10}$$

which is the same as proposed in Refs. [1] and [13]. The cloak is considered to be lossless at the working frequency.

4. Cloak's exterior boundary

In this section, we will prove that no reflection is excited at the exterior boundary. The transmitted electric field $\widehat{\mathbf{E}}^i$ and magnetic field $\widehat{\mathbf{H}}^i$ without interacting with the inner boundary S_2 are expressed as

$$\widehat{\mathbf{E}}^i = g\mathbf{E}^i, \; \widehat{\mathbf{H}}^i = g\mathbf{H}^i, \tag{11}$$

where $\mathbf{E}^i$ and $\mathbf{H}^i$ represent the electric and magnetic fields of the external incident waves. According to Eq. (9), it is easily seen that the fields expressed in Eq. (11) satisfy Maxwell's equations in the cloak medium. Thus in order to prove no reflection excited at S_1, one only needs to confirm that tangential components of $\mathbf{E}^i$ ($\mathbf{H}^i$) and $\widehat{\mathbf{E}}^i$ ($\widehat{\mathbf{H}}^i$) keep continuous across S_1.

Decompose $\widehat{\mathbf{E}}^i$ and $\widehat{\mathbf{H}}^i$ into $\widehat{\mathbf{E}}^i = [\widehat{E}_n^i, \; \widehat{E}_{t_1}^i, \; \widehat{E}_{t_2}^i]$ and $\widehat{\mathbf{H}}^i = [\widehat{H}_n^i, \; \widehat{H}_{t_1}^i, \; \widehat{H}_{t_2}^i]$, where the subscripts "$n$" represent S_1's normal direction pointing outward from the cloak; "t_1" and "t_2" represent S_1's two tangential directions, which are vertical with each

other. Thus Eq. (11) can also expressed as

$$\begin{bmatrix} \widehat{E}_n^i \\ \widehat{E}_{t_1}^i \\ \widehat{E}_{t_2}^i \end{bmatrix} = [\widehat{n}, \ \widehat{t}_1, \ \widehat{t}_2]^{-1} g\mathbf{E^i}, \qquad \begin{bmatrix} \widehat{H}_n^i \\ \widehat{H}_{t_1}^i \\ \widehat{H}_{t_2}^i \end{bmatrix} = [\widehat{n}, \ \widehat{t}_1, \ \widehat{t}_2]^{-1} g\mathbf{H^i}, \qquad (12)$$

where $\widehat{n}, \widehat{t}_1, \widehat{t}_2$ represent the unit vectors in n, t_1, and t_2 directions, respectively.

At the exterior boundary S_1, $q_1 = x$, $q_2 = y$, and $q_3 = z$. So $f_i(q_1, q_2, q_3) - q_i = 0$ $(i = 1, 2, 3)$ characterize the exterior boundary S_1. Therefore, it is obvious that the vectors $\nabla_q f_i - \widehat{C}_i$ $(i = 1, 2, 3)$ lie in the same line as the normal direction n of S_1, where $\widehat{C}_1 = \widehat{x}$, $\widehat{C}_2 = \widehat{y}$, and $\widehat{C}_3 = \widehat{z}$. For the special case when $\nabla_q f_i - \widehat{C}_i = 0$, $\nabla_q f_i - \widehat{C}_i$ can be expressed as $0\widehat{n}$, i.e., the vector with the magnitude 0 in the n direction. Therefore, g on S_1 can be expressed as

$$g = [F_1\widehat{n} + \widehat{x}, \ F_2\widehat{n} + \widehat{y}, \ \widehat{F}_3\widehat{n} + \widehat{z}], \qquad (13)$$

with

$$|F_i| = \sqrt{(\frac{\partial f_i}{\partial q_i} - 1)^2 + (\frac{\partial f_i}{\partial q_j})^2 + (\frac{\partial f_i}{\partial q_k})^2}, \qquad (14)$$

where $i, j, k = 1, 2, 3$ and $i \neq j \neq k$; $F_i = |F_i|$ when the direction of $\nabla_q f_i - \widehat{C}_i$ is as the same as the n direction, and $F_i = -|F_i|$ if the direction of $\nabla_q f_i - \widehat{C}_i$ is opposite to the n direction. Substituting Eq. (13) into Eq. (12) and noticing that $\widehat{n}, \widehat{t}_1$, and $\widehat{t}_2$ are orthogonal with each other, it is easily obtained that at S_1

$$\widehat{E}_{t_1}^i = \mathbf{E}^i \cdot \widehat{t}_1, \ \widehat{H}_{t_1}^i = \mathbf{H}^i \cdot \widehat{t}_1, \qquad (15)$$

$$\widehat{E}_{t_2}^i = \mathbf{E}^i \cdot \widehat{t}_2, \ \widehat{H}_{t_2}^i = \mathbf{H}^i \cdot \widehat{t}_2, \qquad (16)$$

which indicates that the tangential components of $\mathbf{E}^i$ ($\mathbf{H}^i$) and $\widehat{\mathbf{E}}^i$ ($\widehat{\mathbf{H}}^i$) are continuous across S_1. Thus, it is proved that no reflection is excited at the exterior boundary.

Consider the permittivity and permeability at S_1 for the transformed cloak. It should be noticed that no flection excited at the exterior boundary does not imply that the exterior boundary is a perfectly matched layer (PML), where the permittivity and permeability at S_1 have the PML form $\widehat{\overline{\overline{\varepsilon}}} = \widehat{\overline{\overline{\mu}}} = diag[u, 1/u, 1/u]$ with the principle axes in n, t_1 and t_2 directions, respectively. We find that parameters at S_1 have the PML form only when g is a symmetry matrix. Observing Eq. (13), we have $g^T \widehat{t}_1 = \widehat{t}_1$ and $g^T \widehat{t}_2 = \widehat{t}_2$, indicating that $\widehat{t}_1$ and $\widehat{t}_2$ are the eigen vectors of g^T with the same eigen value 1. Considering g is a symmetry matrix, we can know the other eigen vector of g^T is $\widehat{n}$ with eigen value $\det(g)$. Thus, $\widehat{n}, \widehat{t}_1$, and $\widehat{t}_2$ are eigen vectors of $Q^{-1} = (g^T)^{-1}g^{-1}$, with eigen values $1/\det(g)^2$, 1, and 1, respectively. Thus based on Eq. (10), $\widehat{\overline{\overline{\varepsilon}}}$ and $\widehat{\overline{\overline{\mu}}}$ for a symmetric g can be expressed as

$$\widehat{\overline{\overline{\varepsilon}}} = \widehat{\overline{\overline{\mu}}} = diag[\frac{1}{\det(g)}, \ \det(g), \ \det(g)], \qquad (17)$$

where the diagonal elements correspond to the principle axes $\widehat{n}, \widehat{t}_1$, and $\widehat{t}_2$, respectively. The radially transformed cylindrical and spherical cloaks fall into this category [1].

5. Cloak's inner boundary

In this section, we will prove that at the inner boundary S_2, no reflection is excited and no field can penetrate into the cloaked region. As discussed in section III, the inner boundary is constructed by blowing up a line or a point, as seen in Fig. 2(a) and (b). So in the following, two cases that: (1) line transformed cloaks, (2) point transformed cloaks, will be discussed separately.

5.1. Case (1): line transformed cloaks

Assume that $x = b_1(s)$, $y = b_2(s)$ and $z = b_3(s)$ characterize the line, which is mapped to the inner boundary S_2. We have $f_1(q_1, q_2, q_3) = b_1(s)$ and $f_2(q_1, q_2, q_3) = b_2(s)$, and $f_3(q_1, q_2, q_3) = b_3(s)$ at S_2. Each point (b_1, b_2, b_3) on the line maps to a closed curve on S_2. The parameter s can be expressed as a function of q_1, q_2, q_3 with $s = u(q_1, q_2, q_3)$. $\nabla_q s = \partial u/\partial q_1 \widehat{x} + \partial u/\partial q_2 \widehat{y} + \partial u/\partial q_3 \widehat{z}$ is the gradient of s, which points in the direction of the greatest increase rate of s. For $\nabla_q b_i$, we have $\nabla_q b_i = \partial b_i/\partial s \nabla_q s$, where $i = 1, 2, 3$. Thus, $\nabla_q b_i$ and $\nabla_q s$ have the same direction. For the ease of our discussion,
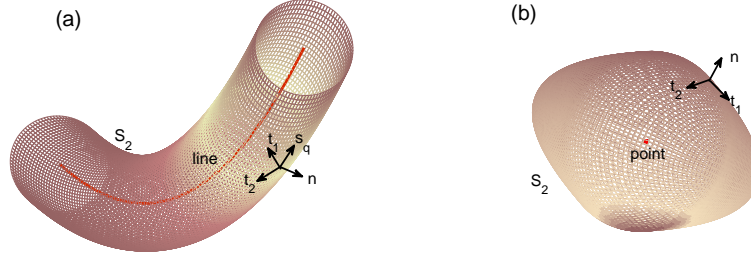

Figure 2. Illustration of cloak's inner boundary for (a) a line transformed cloak; (b) a point transformed cloak.

we again decompose the incident fields at the inner boundary as $\widehat{\mathbf{E}}^i = [\widehat{E}^i_n, \widehat{E}^i_{t_1}, \widehat{E}^i_{t_2}]$, $\widehat{\mathbf{H}}^i = [\widehat{H}^i_n, \widehat{H}^i_{t_1}, \widehat{H}^i_{t_2}]$, where the subscripts "$n$" denotes S_2's normal direction, which points outward from the cloaked region; "t_1" and "t_2" denote the tangential directions of S_2, with t_1 vertical with the plane determined by two vectors in n and $\nabla_q s$ directions, and t_2 vertical with t_1. Since s varies on the surface S_2, the direction of $\nabla_q s$ denoted by s_q should not be parallel to S_2's normal direction. Thus, the plane determined by the vectors in n and $\nabla_q s$ directions always exists. The n, t_1, and t_2 directions are unique, as illustrated in Fig. 2(a). $\widehat{\mathbf{E}}^i$ and $\widehat{\mathbf{H}}^i$ at S_2 can also be expressed in Eq. (12), however, with the different definitions of n, t_1 and t_2. Since $f_i(q_1, q_2, z) - b_i(s) = 0$ $(i = 1, 2, 3)$ characterize the inner boundary S_2, $\nabla_q f_i - \nabla_q b_i$ characterize the normal direction of S_2. Then g at S_2 can be written as

$$g = [F_1\widehat{n} + \nabla_q b_1, \ F_2\widehat{n} + \nabla_q b_2, \ F_3\widehat{n} + \nabla_q b_3], \tag{18}$$

with

$$|F_i| = \sqrt{(\partial f_i/\partial q_1 - \partial b_i/\partial q_1)^2 + (\partial f_i/\partial q_2 - \partial b_i/\partial q_2)^2 + (\partial f_i/\partial q_3 - \partial b_i/\partial q_3)^2}, \tag{19}$$

where $F_i = |F_i|$ when the direction of $\nabla_q f_i - \nabla_q b_i$ is as the same as n direction, and $F_i = -|F_i|$ when the direction of $\nabla_q f_i - \nabla_q b_i$ is opposite to n direction.

Notice that t_1 is orthogonal with both n and s_q, where $\widehat{s}$ denotes the unit the vector in the direction of $\nabla_q s$. Substituting Eq. (18) into Eq. (12), it is easily derive that

$$\widehat{E}_{t1}^i = \widehat{H}_{t1}^i = 0. \tag{20}$$

However, the other components of fields are not zero. In particular,

$$\widehat{E}_{t2}^i = (\widehat{s} \cdot \widehat{t}_2)[B_1,\ B_2,\ B_3]\mathbf{E}^i, \tag{21}$$

$$\widehat{H}_{t2}^i = (\widehat{s} \cdot \widehat{t}_2)[B_1,\ B_2,\ B_3]\mathbf{H}^i, \tag{22}$$

$$\widehat{E}_n^i = [F_1 + B_1(\widehat{s} \cdot \widehat{n}),\ F_2 + B_2(\widehat{s} \cdot \widehat{n}),\ F_3 + B_3(\widehat{s} \cdot \widehat{n})]\mathbf{E}^i, \tag{23}$$

$$\widehat{H}_n^i = [F_1 + B_1(\widehat{s} \cdot \widehat{n}),\ F_2 + B_2(\widehat{s} \cdot \widehat{n}),\ F_3 + B_3(\widehat{s} \cdot \widehat{n})]\mathbf{H}^i, \tag{24}$$

with

$$B_i = \sqrt{\partial b_i/\partial q_1^2 + \partial b_i/\partial q_2^2 + \partial b_i/\partial q_3^2}. \tag{25}$$

To further investigate how the waves interact with the inner boundary, the values of the permittivity and permeability at S_2 are needed. Observing g expressed in Eq. (18), it is easily obtained that $g^T \widehat{t}_1 = 0$, indicating that $Q\widehat{t}_1 = gg^T \widehat{t}_1 = 0$. Thus one of Q's eigen vectors is $\widehat{t}_1$ with the eigen value $\lambda_{t_1} = 0$, implying $\det(g) = 0$. Because Q is a symmetry matrix, the other two eigen vectors denoted by $\widehat{a}$ and $\widehat{b}$ should be orthogonal to each other and in $n - t_2$ plane. The corresponding eigen values are denoted by λ_a and λ_b, respectively, with

$$\lambda_a \lambda_b = |\widehat{n} \times \widehat{s}|^2 |\widehat{F} \times \widehat{B}|^2, \tag{26}$$

where $\widehat{F} = F_1\widehat{x} + F_2\widehat{y} + F_3\widehat{z}$, $\widehat{B} = B_1\widehat{x} + B_2\widehat{y} + B_3\widehat{z}$.

Since $\lambda_{t_1}\lambda_a\lambda_b = \det(Q) = \det(g)^2$, $\lambda_{t_1} = \det(g)^2/(\lambda_a\lambda_b)$. Observing Eq. (10), it is obtained that $Q\overline{\overline{\epsilon}} = Q\overline{\overline{\mu}} = \det(g)$. Therefore, it is known that $\widehat{t}_1$, $\widehat{a}$ and $\widehat{b}$ are the principle axes of the cloaked medium at S_2 with $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ expressed as

$$\overline{\overline{\epsilon}} = \overline{\overline{\mu}} = diag[\lambda_a\lambda_b/\det(g),\ \det(g)/\lambda_a,\ \det(g)/\lambda_b], \tag{27}$$

where the diagonal elements correspond to the principle axes $\widehat{t}_1, \widehat{a}$, and $\widehat{b}$, respectively. Since $\det(g) = 0$, we have $\epsilon_a = \mu_a = \epsilon_b = \mu_b = 0$, indicating the cloaked medium at S_2 is isotropic in $n - t_2$ plane. Therefore, $\widehat{n}$ and $\widehat{t}_2$ can be considered as the principle axes with $\epsilon_n = \mu_n = \epsilon_{t_2} = \mu_{t_2} = 0$. For ϵ_{t_1} and μ_{t_1}, it is seen that they have infinitely large values. Thus, the inner boundary operates similarly as a combination of the PEC (perfect electric conductor) and PMC (perfect magnetic conductor), which can support both electric and magnetic surface displacement currents in t_1 direction [8, 12, 15]. In order to have zero reflection at S_2, the boundary conditions at this PEC and PMC combined layer require that the incident electric (magnetic) fields in t_1 direction and normal electric (magnetic) displacement fields are all zero. From Eq. (20), we have $\widehat{E}_{t_1}^i = \widehat{H}_{t_1}^i = 0$. Since $\epsilon_n = \mu_n = 0$, it is obtained that $\widehat{D}_n^i = \widehat{B}_n^i = 0$. Therefore, it is achieved that no reflection is excited at S_2, and $\widehat{\mathbf{E}}^i$ and $\widehat{\mathbf{H}}^i$ expressed in Eq. (11) are just the total fields in the cloak medium. The PEC and PMC combined layer guarantees that no field can penetrate into the cloaked region. It is worth noting that the induced displacement surface currents in t_1 direction make $\widehat{E}_{t_2}^i$ and $\widehat{H}_{t_2}^i$ at S_2 down to zero. However, $\widehat{E}_{t_2}^i$ and $\widehat{H}_{t_2}^i$ are not zero at the location approaching S_2 in the cloak medium. Thus $\widehat{E}_{t_2}^i$ and $\widehat{H}_{t_2}^i$ are discontinuous across the inner boundary S_2 [8, 12, 15].

5.2. Case (2): point transformed cloaks

In this case, a point with the coordinate $[c_1, c_2, c_3]$ maps to the inner boundary. At S_2, we have $f_1(q_1, q_2, q_3) = c_1$, $f_2(q_1, q_2, q_3) = c_2$, and $f_3(q_1, q_2, q_3) = c_3$. The incident electric and magnetic fields at the inner boundary can be decomposed into $\widehat{\mathbf{E}}^i = [\widehat{E}_n^i, \widehat{E}_{t_1}^i, \widehat{E}_{t_2}^i]$ and $\widehat{\mathbf{H}}^i = [\widehat{H}_n^i, \widehat{H}_{t_1}^i, \widehat{H}_{t_2}^i]$, where the definition of the subscript "n" denotes S_2's normal direction, which direct outward from the cloaked region; "t_1" and "t_2 represent S_2's two tangential directions, which are vertical with each other, as shown in Fig. 2(b). Consider g at the inner boundary, which can be expressed as

$$g = diag[F_1 \widehat{n}, \quad F_2 \widehat{n}, \quad F_3 \widehat{n}], \tag{28}$$

with

$$|F_i| = \sqrt{(\partial f_i / \partial q_1)^2 + (\partial f_i / \partial q_2)^2 + (\partial f_i / \partial q_3)^2}, \tag{29}$$

where $i = 1, 2, 3$. Then substituting Eq. (27) into Eq. (11), we derive that at S_2

$$\widehat{E}_{t_1}^i = \widehat{H}_{t_1}^i = 0, \tag{30}$$

$$\widehat{E}_{t_2}^i = \widehat{H}_{t_2}^i = 0, \tag{31}$$

$$\widehat{E}_n^i = [F_1, \quad F_2, \quad F_3]\mathbf{E}^i, \quad \widehat{H}_n^i = [F_1, \quad F_2, \quad F_3]\mathbf{H}^i. \tag{32}$$

Unlike the case (1), in this case tangential fields are all zero, implying that no field discontinuity exists at S_2.

Analyzing Q similarly as in the case (1), we obtain that $\widehat{n}$ is eigen vector of Q with the eigen value $\lambda_n = F_1^2 + F_2^2 + F_3^2$. While the other two eigen vectors are $\widehat{t}_1$ and $\widehat{t}_2$ with the corresponding eigen values $\lambda_{t1} = \lambda_{t2} = 0$, indicating $\det(g) = 0$. Considering $\lambda_n \lambda_{t1} \lambda_{t2} = \det(g)^2$, we have $\lambda_{t1} = \lambda_{t2} = \det(g)/\sqrt{(F_1^2 + F_2^2 + F_3^2)}$. Therefore, $\widehat{n}$, $\widehat{t}_1$, and $\widehat{t}_2$ are principle axes of the cloak medium at S_1 with $\widehat{\overline{\overline{\varepsilon}}}$ and $\widehat{\overline{\overline{\mu}}}$ given as

$$\widehat{\overline{\overline{\varepsilon}}} = \widehat{\overline{\overline{\mu}}} = diag[\det(g)^2 / (F_1^2 + F^2 + F^3), \quad \sqrt{F_1^2 + F^2 + F^3}, \quad \sqrt{F_1^2 + F^2 + F^3}], \tag{33}$$

where the diagonal elements are in the principle axes $\widehat{n}$, $\widehat{t}_1$, and $\widehat{t}_2$, respectively. Since $\det(g) = 0$, $\epsilon_n = \mu_n = 0$. Considering that $\epsilon_n = \mu_n = 0$ and tangential components of incident fields at S_2 are zero, it can be conclude that no reflection is excited at S_2, and no field penetrates into the cloaked region. The fields expressed in Eq. (11) are the total fields in the cloak medium.

In the above sections, it has been proved that no reflection is excited at both the exterior boundary and the inner boundary of the cloak, and no field can penetrate into the cloaked region. Therefore, the invisibility of invisibility cloaks with arbitrary shape constructed by general coordinate transformations is confirmed.

6. Transformation under arbitrary coordinate system

The cloak parameters and the fields inside the cloak are expressed in Eqs. (10) and (11), respectively. These results are expressed under the Cartesian coordinate system, i.e., (q_1, q_2, q_3) representing Cartesian coordinates in the transformed space. However sometimes, it is much easier to discuss cloaks under other coordinate systems, such as the cylindrical cloak under the cylindrical coordinate system (r, ϕ, z). Thus, it is necessary to obtain the corresponding expressions for cloak parameters and the fields in a cloak under an arbitrary coordinate system, which has also been discussed in Ref. [13].

Consider coordinate transformation, where (q_1, q_2, q_3) denotes the coordinates of an arbitrary coordinate system (u, v, w) in the transformed space. The spatial metric tensor of such coordinate system is $Q_u = g_u g_u^T$, where g_u can be obtained easily by considering the relationship between Cartesian coordinate system and this arbitrary coordinate system. For the cylindrical coordinate system and spherical coordinate system, the metric tensors are $diag[1, r^2, 1]$ and $diag[1, r^2, r^2 sin^2\theta]$, respectively. The spatial metric tensor of (q_1, q_2, q_3) is expressed as $Q_q = g_q g_q^T$, where g_q is shown in Eq. (4). Assuming that (x_1, y_1, z_1) are the corresponding Cartesian coordinates of (q_1, q_2, q_3) in the transformed space, the spatial metric tensor of (x_1, y_1, z_1) is then obtained as $Q_c = g_c g_c^t$, where $g_c = g_{u1}^{-1} g_q$, and g_{u1} represents g_u expressed under the coordinates (q_1, q_2, q_3). Thus, from Eq. (10), the permittivity and permeability of the cloak in Cartesian coordinates are obtained as $\widehat{\overline{\overline{\varepsilon}}} = \widehat{\overline{\overline{\mu}}} = \frac{\det(g_q)}{\det(g_{u1})} g_{u1}^T Q_q^{-1} g_{u1}$. Then expressing $\widehat{\overline{\overline{\varepsilon}}}$ and $\widehat{\overline{\overline{\mu}}}$ in (u, v, w) coordinate system, we easily have

$$\widehat{\overline{\overline{\varepsilon}}} = \widehat{\overline{\overline{\mu}}} = \frac{\det(g_q)}{\det(g_{u1})} P_1 Q_q^{-1} Q_{u1} P_1^{-1}, \tag{34}$$

where $P_1 = diag[p_1{}^1, p_2{}^1, p_3{}^1]$ with $p_i{}^1 = \sqrt{g_{u1\,i1}^2 + g_{u1\,i2}^2 + g_{u1\,i3}^2}$, and $Q_{u1} = g_{u1} g_{u1}^T$. As an example, $P_1 = diag[1, r, 1]$ for the cylindrical coordinate system.

Consider the fields in the cloak. It is easy to know that the fields expressed in Cartesian coordinate system are $\widehat{\mathbf{E}} = g_c \mathbf{E}^i$ and $\widehat{\mathbf{H}} = g_c \mathbf{H}^i$. Thus, the fields expressed in the (u, v, w) coordinate system are expressed as following

$$\widehat{\mathbf{E}} = P_1 Q_{u1}^{-1} g_q (g_{u0})^T P_0^{-1} \mathbf{E}^{i'} \ , \ \widehat{\mathbf{H}} = P_1 Q_{u1}^{-1} g_q (g_{u0})^T P_0^{-1} \mathbf{H}^{i'}, \tag{35}$$

where g_{u0} represents g_u expressed under the coordinates (q_1', q_2', q_3') and $Q_{u0} = g_{u0} g_{u0}^T$, $P_0 = diag[p_1{}^0, p_2{}^0, p_3{}^0]$ with $p_i{}^0 = \sqrt{g_{u0\,i1}^2 + g_{u0\,i2}^2 + g_{u0\,i3}^2}$; $\mathbf{E}^{i'}$ and $\mathbf{H}^{i'}$ represent incident electrical and magnetic field vectors expressed under the (u, v, w) coordinate system.

If (q_1', q_2', q_3') denotes the corresponding coordinates of the coordinate system (u, v, w) in the original space, then g_q can be written as $g_q = g_s g_{u0}$, where

$$g_s = \begin{bmatrix} \frac{\partial q_1'}{\partial q_1'} & \frac{\partial q_2'}{\partial q_1'} & \frac{\partial q_3'}{\partial q_1'} \\ \frac{\partial q_1'}{\partial q_2'} & \frac{\partial q_2'}{\partial q_2'} & \frac{\partial q_3'}{\partial q_2'} \\ \frac{\partial q_1'}{\partial q_3} & \frac{\partial q_2'}{\partial q_3} & \frac{\partial q_3'}{\partial q_3} \end{bmatrix}. \tag{36}$$

Then Eqs. (34) and (35) can be expressed as

$$\widehat{\overline{\overline{\varepsilon}}} = \widehat{\overline{\overline{\mu}}} = \frac{\det(g_s) \det(g_{u0})}{\det(g_{u1})} P_1 (g_s^T)^{-1} Q_{u0}^{-1} g_s^{-1} Q_{u1} P_1^{-1}, \tag{37}$$

$$\widehat{\mathbf{E}} = P_1 Q_{u1}^{-1} g_s (Q_{u0})^T P_0^{-1} \mathbf{E}^{i'} \ , \ \widehat{\mathbf{H}} = P_1 Q_{u1}^{-1} g_s (Q_{u0})^T P_0^{-1} \mathbf{H}^{i'}. \tag{38}$$

7. Examples: cylindrical and spherical cloaks

In this section, based on the results obtained above, the well known radially transformed cylindrical and spherical cloaks will be discussed as examples.

7.1. Cylindrical cloaks

A two-dimensional cylindrical cloak is constructed by compressing EM fields in a cylindrical region $r^{'} < b$ into a concentric cylindrical shell $a < r < b$. Its inner boundary is blown up by a straight line. Thus, a cylindrical cloak is actually a line transformed cloak. Here consider a generalized coordinate transformation that $r^{'} = f(r)$ with $f(a) = 0$ and $f(b) = b$, while θ and z are kept unchanged. Thus, Q_{u0} and Q_{u1} defined in the above section are $diag[1, f(r)^2, 1]$ and $diag[1, r^2, 1]$, respectively, which indicates that $\det(g_{u0}) = f(r)$ and $\det(g_{u1}) = r$. P_0 and P_1 are $diag[1, f(r), 1]$ and $diag[1, r, 1]$, respectively. g_s is equal to $diag[f^{'}(r), 1, 1]$. Substituting these expressions into Eq. (37), the permittivity and permeability of the cloak expressed in cylindrical coordinate system are obtained easily

$$\epsilon_r = \mu_r = \frac{f(r)}{rf^{'}(r)}, \epsilon_\theta = \mu_\theta = \frac{rf^{'}(r)}{f(r)}, \epsilon_z = \mu_z = \frac{f(r)f^{'}(r)}{r}, \tag{39}$$

It is seen that at the exterior boundary $r = b$, the cloak medium has the PML form with $\epsilon_r = \mu_r = 1/f^{'}(b)$ and $\epsilon_\theta = \mu_\theta = \epsilon_z = \mu_z = f^{'}(b)$, which results from the symmetry of g_c, which can be calculated easily.

Consider the fields $\mathbf{E}^i$ and $\mathbf{H}^i$ incident upon the cloak. It is derived that $P_1 Q_{u1}^{-1} g_s (Q_{u0})^T P_0^{-1} = diag[f^{'}(r), f(r)/r, 1]$. Then, the fields in the cloaked medium can be obtained directly from Eq. (38)

$$\widehat{E}_r(r, \theta, z) = f^{'}(r) E_r^i(f(r), \theta, z), \ \widehat{H}_r(r, \theta, z) = f^{'}(r) H_r^i(f(r), \theta, z), \tag{40}$$

$$\widehat{E}_\theta(r, \theta, z) = \frac{f(r)}{r} E_\theta^i(f(r), \theta, z), \ \widehat{H}_\theta(r, \theta, z) = \frac{f(r)}{r} H_\theta^i(f(r), \theta, z), \tag{41}$$

$$\widehat{E}_z(r, \theta, z) = E_z^i(f(r), \theta, z), \ \widehat{H}_z(r, \theta, z) = H_z^i(f(r), \theta, z), \tag{42}$$

where $[E_r^i, E_\theta, E_z]$ and $[H_r^i, H_\theta, H_z]$ are components of the incident fields expressed in cylindrical coordinate system. When $f(r) = b(r - a)/(b - a)$, substituting the expression of $f(r)$ into Eqs. (40)-(42), we obtain the fields in the cloak medium, which is just the result in Ref. [6].

At the inner boundary, one can easily see that s_q and t_2 are both in the z direction, t_1 is in the θ direction, and n is in the r direction. Therefore, no matter what $f(r)$ is, ϵ_θ and μ_θ are infinitely large, and other components are zero, as seen in Eq. (39). E_θ and H_θ, D_r and B_r are all zero at S_2, which guarantees that no reflection is excited at S_2 as analyzed above. The surface displacement currents are induced to make E_z and H_z down to zero at S_2. While E_z and H_z are not zero at the locations approaching S_2 in the cloak medium. Thus, E_z and H_z are discontinuous across the inner boundary [8, 12, 15].

7.2. Spherical Cloaks

A three-dimensional spherical cloak can be constructed by compressing EM fields in a spherical region $r^{'} < b$ into a spherical shell $a < r < b$. Its inner boundary is blown up by a point. Thus, a spherical cloak is actually a point transformed cloak. Here a generalized radial coordinate transformation that $r^{'} = f(r)$ with $f(a) = 0$ and $f(b) = b$, is considered. Similarly as in the above example of the cylindrical cloak, the permittivity and permeability of the spherical cloak expressed in spherical coordinate system are derived

$$\epsilon_r = \mu_r = \frac{f(r)^2}{r^2 f^{'}(r)}, \epsilon_\theta = \mu_\theta = \epsilon_\phi = \mu_\phi = f^{'}(r). \tag{43}$$

At the exterior boundary $r = b$, the cloak medium has the PML form with $\epsilon_r = \mu_r = 1/f^{'}(b)$ and $\epsilon_\theta = \mu_\theta = \epsilon_\phi = \mu_\phi = f^{'}(b)$, due to the symmetry of g_c.

Consider the incident fields $\mathbf{E}^i = [E^i_r, \ E^i_\theta, E^i_\phi]$ and $\mathbf{H}^i = [H^i_r, \ H^i_\theta, \ H^i_\phi]$ incident upon the cloak, from Eq. (39), the fields in the cloaked medium are obtained directly

$$\widehat{E}_r(r,\theta,\phi) = f^{'}(r)E^i_r(f(r),\theta,\phi), \ \widehat{H}_r(r,\theta,\phi) = f^{'}(r)H^i_r(f(r),\theta,\phi), (44)$$

$$\widehat{E}_\theta(r,\theta,\phi) = \frac{f(r)}{r}E^i_\theta(f(r),\theta,\phi), \ \widehat{H}_\theta(r,\theta,\phi) = \frac{f(r)}{r}H^i_\theta(f(r),\theta,\phi), (45)$$

$$\widehat{E}_\phi(r,\theta,\phi) = \frac{f(r)}{r}E^i_\phi(f(r),\theta,\phi), \ \widehat{H}_\phi(r,\theta,\phi) = \frac{f(r)}{r}H^i_\phi(f(r),\theta,\phi).(46)$$

When $f(r) = b(r - a)/(b - a)$, the fields obtained from the above equations agrees with the results in Ref. [5]. However, the process of the calculation here is simpler.

At the inner boundary, observing Eqs. (45) and (46), the tangential components of fields are zero [5, 15, 16]. Combining with $\epsilon_n = \mu_n = 0$, it is known that no field can penetrate into the cloaked region.

8. Conclusions

In this paper, we have studied the properties of invisibility cloaks constructed by general coordinate transformations. The invisibility of cloaks is confirmed, by proving that no reflection is excited at both the exterior and interior boundaries of the cloak, and no field can penetrate into the cloaked region. The fields in the cloak medium are related to the fields in original EM space through $\widehat{\mathbf{E}} = g\mathbf{E}^i$ and $\widehat{\mathbf{H}} = g\mathbf{H}^i$. Therefore, to calculate fields in the cloak medium, there is no need to process tedious calculations from the complex material parameters. At the exterior boundary, when g is a symmetry matrix, the permittivity and permeability of the cloak medium have the PML form, which is just the case for our well known radially transformed cylindrical and spherical cloaks. At the interior boundary, the properties of the cloak for line and point transformed invisibility cloaks are quite different. For a line transformed cloak, the components of the permittivity and permeability in t_1 direction (defined in section V) are infinitely large. While the other components are all zero. The fields in t_2 direction are discontinuous across the inner boundary. The surface displacement currents in t_1 direction are induced to make this discontinuity self-consistent. For any point transformed cloak, at the inner boundary, the components of the permittivity and permeability don't have infinitely large component, and the permittivity and permeability in the normal direction are zero. The tangential fields at the inner boundary are zero, implying no discontinuity exist. Therefore, comparing to line transformed cloaks, point transformed cloaks are more practical due to the absence of the singularity of the cloak medium.

Acknowledgements

References

[1] J. B. Pendry, D. Schurig, and D. R. Smith 2006 *Science* **312** 1780
[2] U. Leonhardt 2006 *Science* **312** 1777
[3] S. A. Cummer, B. I. Popa, D. Schurig, D. R. Smith, and J. B. Pendry 2006 *Phys. Rev. E.* **74** 036621
[4] F. Zolla, S. Guenneau, A. Nicolet, and J. B. Pendry 2007 *Opt. Lett.* **32**, 1069
[5] H. S. Chen, B. I. Wu, B. L. Zhang, and J. A. Kong 2007 *Phys. Rev. Lett.* **99** 063903
[6] Z. C. Ruan, M. Yan, C. W. Neff, and M. Qiu 2007 *Phys. Rev. Lett.* **99** 113903
[7] D. Schurig, J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith 2006 *Science* **314** 977
[8] B. L. Zhang, H. S. Chen, B. I. Wu, Y. Luo, L. X. Ran, and J. A. Kong 2007 *Phys. Rev. B* **76** 121101
[9] M. Rahm, D. Schurig, D. A. Roberts, S. A. Cummer, D. R. Smith and J. B. Pendry 2007 http: //www.arXiv:0706.2452v1[physics.optics]
[10] H. Y. Chen and C. T. Chan 2007 *Appl. Phys. Lett.* **90** 241105
[11] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann 2007 *Phys. Rev. Lett.* **99** 183901
[12] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann 2007 *Opt. Express* **15** 12717
[13] U. Leonhardt, T. G. Philbin 2006 *New J. Phys.* **8** 247
[14] U. Leonhardt 2006 *New J. Phys.* **8** 118
[15] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann 2007 *Comm. Math. Phys.* **275** 749
[16] R. Webber 2007 http: //www.arXiv:0711.0507[physics.optics]